MATHEMATICS

Classes of completely distributive complete lattices

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1. INTRODUCTION

The main objective of this paper is to study certain classes of completely distributive complete lattices and their complete lattices of complete congruence relations. An important tool in our investigations is a fundamental result obtained by Raney [6], [7], which states that a complete lattice is completely distributive if and only if it is a subdirect product of complete chains. In section 3 we start our work with investigating the lattice of complete congruence relations of a complete chain. The main result of this section is that this lattice is a Heyting algebra and therefore distributive. (Theorem 3.6 and Corollary 3.7). In section 4 we prove some theorems which pave the way to our work in section 5. The main part of section 5 is devoted to the class $\mathscr{R}$ which consists of all completely distributive complete lattices, all of whose complete homomorphic images are complete rings of sets. It is well known that every completely distributive complete lattice is a complete homomorphic image of a complete ring of sets but not every such lattice is a complete ring of sets. Examples of lattices that belong to $\mathscr{R}$ are complete atomic Boolean algebras and complete (dual) ordinals. Indeed, every complete atomic Boolean algebra is a power algebra.

Also, a complete (dual) ordinal is a complete ring of sets and a complete homomorphic image of a complete (dual) ordinal is again a complete (dual) ordinal. As our first result we prove that every completely distributive complete lattice which is countable belongs to $\mathscr{R}$ (Theorem 5.1). We then show that a completely distributive complete lattice belongs to $\mathscr{R}$ if and only if its lattice of complete congruence relations is a power algebra. We finally consider a class which arises in a natural way in connection with $\mathscr{R}$. That is the class $\mathscr{D}$ of all completely distributive complete lattices none of whose complete homomorphic images (except the one element lattice) is a complete ring of sets. We show that $\mathscr{D}$ consists precisely of those completely distributive complete lattices which are dense in itself. (Theorem 5.7).

2. PRELIMINARIES

If L is a lattice and $a, b \in L$, $S \subseteqq L$, then $a+b$, ab and $\sum S$, $\prod S$ will denote sums (joins) and products (meet) respectively. For $a, b \in L$, $a \leqslant b$, $[a, b]_L$, $(a, b)_L$, $[a, b)_L$, $(a]_L$ and $[a)_L$ have the usual meaning. In most cases if there is no danger of confusion, we will simply write $[a, b]$ instead of $[a, b]_L$ etc.

A complete lattice is *completely distributive* if it satisfies

$$\prod_{s \in S} \sum_{t \in T} x_{st} = \sum_{\varphi \in T^S} \prod_{s \in S} x_{s\varphi(s)}.$$

A *complete homomorphism* is a mapping between complete lattices which preserves finite and infinite sums and products. In this paper the notion of subdirect product of complete lattices is used in the usual sense, but it is understood that the corresponding embedding and projections are complete. If L is a complete lattice then a *complete congruence relation* θ on L is a congruence relation which enjoys the substitution property for infinite lattice operations. The complete lattice of complete congruence relations of L is denoted by Con (L). For $\theta \in \text{Con}\,(L)$, $L/_\theta$ denotes the complete quotient lattice of L modulo θ and for $a \in L$, $[a]_\theta$ denotes the congruence class of L modulo θ containing the element a. If $h: L \to L'$ is a complete homomorphism between complete lattices then the *kernel* of h, denoted by ker h is the complete congruence relation θ on L defined by $x \equiv y(\theta) \Leftrightarrow h(x) = h(y)$ for $x, y \in L$. We will use the symbols **1** and **2** to denote the one – and two elements lattice respectively. We finally note that products in Con (L) are the same as products in the lattice of all congruence relations of L. However, sums in Con (L) are in general not the same as sums in the lattice of all congruence relations of L. For concepts used in this paper and not defined, we refer the reader to [2] and [4].

3. THE LATTICE OF COMPLETE CONGRUENCE RELATIONS OF A COMPLETE CHAIN

The main purpose of this section is to prove that if C is a complete chain, then the lattice Con (C) of complete congruence relations of C is a Heyting algebra. We will also show that Con (C) is not necessarily a Boolean algebra.

If L is a complete lattice then the congruence classes of a complete congruence relation are closed intervals. If in particular, L is a complete chain C then every partition of C in closed intervals induces a complete congruence relation, as can be easily seen. We first recall the following theorem which holds for universal algebras with finitary and possibly infinitary operations but which, for convenience, will be formulated for complete lattices.

THEOREM 3.1. Let L be a complete lattice. Let $\theta_1, \theta_2 \in \mathrm{Con}\,(L)$, $\theta_1 \leqslant \theta_2$. Let $\bar{\theta}_2 \in L/_{\theta_1}$ be defined by $[x]_{\theta_1} \equiv [y]_{\theta_1}(\overline{\theta_2}) \Leftrightarrow x \equiv y(\theta_2)$, for $x, y \in L$. Then $[\theta_1, \theta_2]_{\mathrm{Con}\,(L)} \cong [0, \overline{\theta_2}]_{\mathrm{Con}\,(L/\theta_1)}$.

THEOREM 3.2. Let C be a complete chain and suppose $\theta_0 \in \mathrm{Con}\,(C)$. Let $(C_j)_{j \in J}$ be the set of congruence classes of C modulo θ_0. Then

$$[0, \theta_0]_{\mathrm{Con}\,(C)} \cong \underset{j \in J}{X} \mathrm{Con}\,(C_j).$$

PROOF. Define for $\theta \in [0, \theta_0]_{\mathrm{Con}\,(C)}$ and for each $j \in J$, $\theta_j \in \mathrm{Con}\,(C_j)$ by $x \equiv y(\theta_j) \Leftrightarrow x \equiv y(\theta)$ for $x, y \in C_j$. Now define

$$f\colon [0, \theta_0]_{\mathrm{Con}\,(C)} \to \underset{j \in J}{X} \mathrm{Con}\,(C_j)$$

by $(f(\theta))_j = \theta_j$ for $\theta \in [0, \theta_0]_{\mathrm{Con}\,(C)}$ and for each $j \in J$. It is easy to see that f is an isomorphism.

LEMMA 3.3. Let C be a complete chain. Then Con (C) is pseudo-complemented.

PROOF. Let $\theta \in \mathrm{Con}\,(C)$. For each $c \in C$, let $[\underline{c}, \bar{c}] = [c]_\theta$. We now define for each $c \in C$ elements $\tilde{c}$ and $\underset{\sim}{c}$ of C as follows:

(1) if $c < \bar{c}$, then $\tilde{c} = c$;
(2) if $c = \bar{c}$, then $\tilde{c} = \prod \{\underline{x} : \underline{x} < \bar{x}, \underline{x} \geqslant c, x \in C\}$ and dually for $\underset{\sim}{c}$.

Note that $\underset{\sim}{c} \leqslant c \leqslant \tilde{c}$ for each $c \in C$. We now proceed in steps and prove:

(i) For $c, c' \in C$, $c' \in [\underset{\sim}{c}, \tilde{c}]$ we have $\widetilde{c'} = \tilde{c}$ and $\underset{\sim}{c'} = \underset{\sim}{c}$. It suffices to prove that $\widetilde{c'} = \tilde{c}$ and we may assume $\underset{\sim}{c} < \tilde{c}$ and $c \neq c'$. We have the following cases:

(i)$_1$ $c < c' < \tilde{c}$. Since $c < \tilde{c}$, it follows from (1) that $c = \bar{c}$. Also, $c' = \overline{c'}$ since if $c' < \overline{c'}$ then $c = \bar{c} < c' \Rightarrow c = \bar{c} < \underline{c}' \leqslant c' < \overline{c'} \Rightarrow \tilde{c} \leqslant \underline{c}' \leqslant c' < \tilde{c}$. Contradiction. Hence $c' = c'$ and thus by hypothesis and by (2), $\widetilde{c'} = \tilde{c}$.

(i)$_2$ $c' = \tilde{c}$. If $c' < \overline{c'}$ then by (1), $\widetilde{c'} = c' = \tilde{c}$. If $c' = \overline{c'}$ then again by (2) and since $c < c' = \tilde{c}$, we have $\widetilde{c'} = \tilde{c}$.

(i)$_3$ $c \leqslant c' < c$. By the dual of (2) we have $\underline{c} = c$. Also $c' = \overline{c'}$. Indeed, suppose $c' < \widetilde{c'}$. But then $c' < c = \underset{\sim}{c} \Rightarrow c' < \underset{\sim}{c} \Rightarrow \underset{\sim}{c'} \leqslant c' < \widetilde{c'} < \underset{\sim}{c} = c \Rightarrow$ (by the dual of (2)) $\underset{\sim}{c} \geqslant \widetilde{c'} > c' \Rightarrow \underset{\sim}{c} > c'$. Contradiction. Hence $c' = \widetilde{c'}$. Thus we have by (2), $\widetilde{c'} = \{\underset{\sim}{x} : \underset{\sim}{x} < \tilde{x}, \underset{\sim}{x} \geqslant c', x \in C\}$. Now if $x \in C$ and $\underset{\sim}{x} < \tilde{x}$, $\underset{\sim}{x} \geqslant c'$, then $\underset{\sim}{x} \geqslant c$. Indeed, suppose $\underset{\sim}{x} < c$. Then $c' \leqslant \underset{\sim}{x} < c = \underset{\sim}{c} \Rightarrow c' \leqslant \underset{\sim}{x} < \tilde{x} < c \Rightarrow$ (by the dual of (2)) $\tilde{x} \leqslant \underset{\sim}{c} \Rightarrow c' \leqslant \underset{\sim}{x} < \tilde{x} \leqslant \underset{\sim}{c} \Rightarrow c' < \underset{\sim}{c}$. Contradiction. Thus $\underset{\sim}{x} \geqslant c$ and it follows that $\widetilde{c'} = \tilde{c}$.

(ii) If $c, c' \in C$ and $[\underset{\sim}{c}, \tilde{c}] \cap [\underset{\sim}{c'}, \widetilde{c'}] \neq \emptyset$ then $[\underset{\sim}{c}, \tilde{c}] = [\underset{\sim}{c'}, \widetilde{c'}]$. By hypothesis, there exists $x \in C$ such that $x \in [\underset{\sim}{c}, \tilde{c}] \cap [\underset{\sim}{c'}, \widetilde{c'}]$. By (i), $\underset{\sim}{c} = \underset{\sim}{x} = \underset{\sim}{c'}$ and $\tilde{c} = \tilde{x} = \widetilde{c'}$. It follows from (i) and (ii) that the set of closed intervals $\{[\underset{\sim}{c}, \tilde{c}] : c \in C\}$ forms a partition of C which therefore induces a complete congruence relation $\theta^* \in \operatorname{Con}(C)$, such that $[c]_{\theta^*} = [\underset{\sim}{c}, \tilde{c}]$ for $c \in C$.

(iii) $\theta\theta^* = 0$. Suppose $\theta\theta^* \neq 0$. Then there exist $c, c' \in C$, $c \not\equiv c'$, such that $c \equiv c'(\theta)$ and $c \equiv c'(\theta^*)$. We may assume $c < c'$. Thus $c < c' \leqslant \widetilde{c'} = \tilde{c}$. Then by (1), $c = \tilde{c}$. But $\tilde{c} = \widetilde{c'}$, thus $c = \widetilde{c'}$ contradicting $c < c' \leqslant \widetilde{c'}$.

(iv) If $\theta_1 \in \operatorname{Con}(C)$ and $\theta\theta_1 = 0$ then $\theta_1 \leqslant \theta^*$. Suppose $\theta_1 \nleqslant \theta^*$. Then there exist $c, c' \in C$ such that $c \equiv c'(\theta_1)$ and $c \not\equiv c'(\theta^*)$. We may assume $c < c'$. Now $\tilde{c} < c'$ since $c' \leqslant \tilde{c} \Rightarrow \underset{\sim}{c} \leqslant c < c' \leqslant \tilde{c} \Rightarrow c \equiv c'(\theta^*)$. Contradiction. Therefore $\tilde{c} < c'$. Let $[c]_{\theta_1} = [c']_{\theta_1} = [a, b]$. We have $a \leqslant c \leqslant \tilde{c} < c' \leqslant b$ and we consider two cases:

$(\text{iv})_1$ $c < \tilde{c}$. If $\tilde{c} \leqslant b$ then $c \equiv \tilde{c}(\theta_1)$ but $c \equiv \tilde{c}(\theta)$ and $c < \tilde{c}$, so $\theta\theta_1 \neq 0$ but $\theta\theta_1 = 0$ and thus $\tilde{c} > b$. But then $\underset{\sim}{c} \leqslant c < b < \tilde{c} \Rightarrow c \equiv b(\theta)$. Also $c \equiv b(\theta_1)$ and $c < b$ so $\theta\theta_1 \neq 0$. Contradiction.

$(\text{iv})_2$ $c = \tilde{c}$. Then by (2) $\tilde{c} = \prod \{\underset{\sim}{x} : \underset{\sim}{x} < \tilde{x}, \underset{\sim}{x} \geqslant c, x \in C\}$. But $\tilde{c} < b$. Hence there exists $x \in C$, $\underset{\sim}{x} < \tilde{x}$, $\tilde{c} \leqslant \underset{\sim}{x} < b$. Suppose $\tilde{x} \leqslant b$, then $a \leqslant c \leqslant \tilde{c} \leqslant \underset{\sim}{x} \leqslant \tilde{x} \leqslant b \Rightarrow$ $\Rightarrow \underset{\sim}{x} \equiv \tilde{x}(\theta_1)$. But also $\underset{\sim}{x} \equiv \tilde{x}(\theta)$ and $\underset{\sim}{x} < \tilde{x}$, thus $\theta\theta_1 \neq 0$. It follows that $\tilde{x} \nleqslant b$. But $\tilde{x} > b \Rightarrow \underset{\sim}{x} < b < \tilde{x} \Rightarrow \underset{\sim}{x} \equiv b(\theta)$. Also $a \leqslant c \leqslant \underset{\sim}{x} < b \Rightarrow \underset{\sim}{x} \equiv b(\theta_1)$ and again it would follow that $\theta\theta_1 \neq 0$. Thus $\tilde{x} \ngtr b$ which together with $\tilde{x} \nleqslant b$ yields a contradiction.

It follows from (iii) and (iv) that θ^* is the pseudocomplement of θ, completing the proof of the lemma.

LEMMA 3.4. Let C be a complete chain. Then $\operatorname{Con}(C)$ is relatively pseudocomplemented.

PROOF. Let $\theta_1, \theta_2 \in \operatorname{Con}(C)$, $\theta_1 \leqslant \theta_2$. We must prove that $[\theta_1, \theta_2]_{\operatorname{Con}(C)}$ is pseudocomplemented. Let $\overline{\theta_2} \in \operatorname{Con}(C)$ be defined as in Theorem 3.1. Then we have by this theorem, $[\theta_1, \theta_2]_{\operatorname{Con}(C)} \cong [0, \overline{\theta_2}]_{\operatorname{Con}(C/_{\theta_1})}$. But $C/_{\theta_1}$ is a complete chain and therefore $\operatorname{Con}(C/_{\theta_1})$ is pseudocomplemented. But a principal ideal in a pseudocomplemented lattice is again pseudocomplemented. This proves that $[\theta_1, \theta_2]_{\operatorname{Con}(C)}$ is pseudocomplemented.

LEMMA 3.5. Let C be a complete chain. Then complementation in $\operatorname{Con}(C)$ is unique.

PROOF. Suppose $\theta, \theta_1 \in \operatorname{Con}(C)$, $\theta + \theta_1 = 1$, $\theta\theta_1 = 0$. We will show that $\theta_1 = \theta^*$, where θ^* is the pseudocomplement of θ. Let for $c \in C$, $[c]_\theta = [\underset{\sim}{c}, \tilde{c}]$

and $[c]_{\theta_1}=[\underset{\sim}{c}, \tilde{c}]$. It follows from the proof of Lemma 3.3 (cf. (1) and (2)) that we must prove the following. If $c \in C$, then

(3) if $c<\bar{c}$ then $c=\tilde{c}$

(4) if $c=\bar{c}$ then $\tilde{c}=\prod \{\underline{x} : \underline{x}<\bar{x}, \underline{x}\geqslant c, x \in C\}$ and dually.

It suffices to prove (3) and (4). To prove (3) suppose $c<\tilde{c}$. Now $c \in [\underline{c}, \bar{c}] \cap [\underset{\sim}{c}, \tilde{c}]$. Also $\bar{c}\tilde{c}\leqslant\bar{c}$ and $\underline{c}\leqslant c\leqslant\tilde{c} \Rightarrow \underline{c}\leqslant\tilde{c} \Rightarrow \underline{c}\leqslant\bar{c}\tilde{c}$. Thus $\bar{c}\tilde{c} \in [\underline{c}, \bar{c}]$. Similarly $\bar{c}\tilde{c} \in [\underset{\sim}{c}, \tilde{c}]$ and thus $\bar{c}\tilde{c} \in [\underline{c}, \bar{c}] \cap [\underset{\sim}{c}, \tilde{c}]$. It follows that $c \equiv \bar{c}c(\theta\theta_1)$ and hence $c=\bar{c}\tilde{c}$. But $\bar{c}\tilde{c}=\bar{c}$ or $\tilde{c}$, so $c=\bar{c}$ or $\tilde{c}$. Contradiction. For the proof of (4) let $b=\prod \{\underline{x} : \underline{x}<\bar{x}, \underline{x}\geqslant c, x \in C\}$. Thus we must show that $\tilde{c}=b$. We proceed in steps and prove:

(i) if $x \in (c, b)$, then $\underline{x}=\bar{x}=x$. Indeed, assume first that $\underline{x}<x$ then $c=\bar{c}<x \Rightarrow c=\bar{c}<\underline{x}<x\leqslant\bar{x}$. Thus by definition of b, $b\leqslant\underline{x}<x$ but $x<b$. Contradiction. Next, assume $x<\bar{x}$. Then $c=\bar{c}<x \Rightarrow \bar{c}<\underline{x}\leqslant x<\bar{x}$ and thus $b\leqslant\underline{x}\leqslant x$ but $x<b$. Contradiction. It follows that $\underline{x}=\bar{x}=x$.

(ii) $b=\tilde{b}$. Suppose $\underset{\sim}{b}<\tilde{b}$. By definition of b, there exists $x \in C$ such that $b\leqslant\underline{x}<\tilde{b}$, $\underline{x}<\bar{x}$, $\underline{x}\geqslant c$. We have two cases:

(ii)$_1$ $\bar{x}\leqslant\tilde{b}$. Then $\underset{\sim}{b}\leqslant b\leqslant\underline{x}<\bar{x}\leqslant\tilde{b}$ and thus $\underline{x} \equiv \bar{x}(\theta_1)$ and also $\underline{x} \equiv \bar{x}(\theta)$. But this contradicts $\theta\theta_1=0$ since $\underline{x}\neq\bar{x}$.

(ii)$_2$ $\bar{x}\geqslant\tilde{b}$. Then $\underset{\sim}{b}\leqslant b\leqslant\underline{x}<\tilde{b}\leqslant\bar{x}$ and thus $\underline{x} \equiv \tilde{b}(\theta_1)$ and also $\underline{x} \equiv \tilde{b}(\theta)$. But this again contradicts $\theta\theta_1=0$ since $\underline{x}\neq\tilde{b}$.

(iii) $\tilde{c}=b$. Thus by (ii), we must prove $\tilde{c}=b=\tilde{b}$. By definition of b, $c\leqslant b$ thus $c\leqslant\tilde{b}=b$ and hence $\tilde{c}\leqslant\tilde{b}$. Suppose $\tilde{c}<\tilde{b}=b$. Then $\tilde{c}<\underset{\sim}{b}$. Let $\theta' \in \mathrm{Con}\,(C)$ be defined by the set consisting of the following closed intervals: $(\tilde{c}]$; the congruence classes modulo θ_1 which are contained in $(\tilde{c}, \underset{\sim}{b})$, and by $[\underset{\sim}{b})$. Since $\tilde{c}<\underset{\sim}{b}$ we have $\tilde{c} \not\equiv b(\theta')$ and thus $\theta'<1$. We show that $\theta\leqslant\theta'$. Thus we must show that if x is an element of C, then $\underline{x} \equiv \bar{x}(\theta')$. We have three cases:

(iii)$_1$ $\underline{x} \in (\tilde{c}]$. We must show that $\bar{x} \in (\tilde{c}]$. First assume $\underline{x}<c=\bar{c}$. Then $\bar{x}\leqslant\bar{c}(=c)\leqslant\tilde{c}$ so $\bar{x}\leqslant\tilde{c}$. Next, assume $c=\bar{c}\leqslant\underline{x}\leqslant\tilde{c}$. Suppose $\bar{x}>\tilde{c}$ then $c=\bar{c}\leqslant\underline{x}\leqslant\tilde{c}<\bar{x}$ and thus by definition of b, $b\leqslant\underline{x}<\tilde{c}$ or $\tilde{b}\leqslant\tilde{c}$, but $\tilde{c}<\tilde{b}$. Contradiction. Hence $\bar{x}\leqslant\tilde{c}$.

(iii)$_2$ $\underline{x} \in (\tilde{c}, \underset{\sim}{b})$. We must show that $\underline{x}=\bar{x}$. We have $c\leqslant\tilde{c}<\underline{x}<\underset{\sim}{b}\leqslant b$ and thus $\underline{x} \in (c, b)$ and by (i), $\underline{x}=\bar{x}$.

(iii)$_3$ $\underline{x} \in [\underset{\sim}{b})$. To show $\bar{x} \in [\underset{\sim}{b})$, but this is trivial.

It follows that $\theta\leqslant\theta'$ and since obviously $\theta_1\leqslant\theta'$ we have $\theta+\theta_1\leqslant\theta'<1$. Contradiction. This completes the proof of the lemma.

THEOREM 3.6. Let C be a complete chain. Then Con (C) is distributive.

PROOF. We prove that relative complementation in Con (C) is unique. Thus suppose $\theta_1, \theta_2 \in \mathrm{Con}\,(C)$, $\theta_1<\theta_2$. It follows from Theorems 3.1 and 3.2 that $[\theta_1, \theta_2]_{\mathrm{Con}\,(C)} \cong X_{j\in J}\,\mathrm{Con}\,(C_j)$, where $(C_j)_{j\in J}$ is a set of complete chains. By Lemma 3.5, in each Con (C_j) complementation is unique. Hence complementation in $[\theta_1, \theta_2]_{\mathrm{Con}\,(C)}$ is unique.

COROLLARY 3.7. Let C be a complete chain. Then Con (C) is a Heyting algebra.

PROOF. By Theorem 3.6 and Lemma 3.4, Con (C) is a distributive relatively pseudocomplemented lattice. Therefore Con (C) is a Heyting algebra (cf. [2]).

REMARK. It also follows that Con (C) satisfies the infinite distributive law:

$$\theta \sum_{i \in I} \theta_i = \sum_{i \in I} \theta\theta_i.$$

The question arises whether Con (C), C a complete chain, is necessarily a Boolean algebra. The following example shows that the answer is in the negative. (The author is indebted to A. Bousfield and J. Berman for helpful discussions which led to this example). Let $I=[0, 1]$ be the real unit interval and assume that each element of I is represented by its ternary expansion. Consider the class $\mathscr{I}$ of closed intervals $[x, y]$, where $x=0 \cdot x_1x_2 \dots x_k1$, $y=0 \cdot x_1x_2 \dots x_k2$ with $x_i \in \{0, 2\}$ for $1 < i < k$. (Thus the intervals (x, y) are the middle third intervals used in the usual construction of the Cantor discontinuum). Let θ be the complete congruence relation on I defined by the intervals belonging to $\mathscr{I}$ and the closed intervals $[x, x]$, $x \in I \sim \cup \mathscr{I}$. We will show that the pseudocomplement θ^* of θ is 0. Using the notation of the proof of Lemma 3.3, we must prove that for $c \in C$, $\underset{\sim}{c}=\tilde{c}=c$. If $c \in (x, y)$ for some $[x, y] \in \mathscr{I}$ then it follows immediately from (1) and from its dual that $c=\underset{\sim}{c}=\tilde{c}$. If $c=x$ for some $[x, y] \in \mathscr{I}$ then $c=x<y=\tilde{c}$ so again by (1), $\tilde{c}=c$. In order to prove that in this case $\underset{\sim}{c}=c$ note that $\underset{\sim}{c}=x=c$, thus we must use the dual of (2). Assume $x=0 \cdot x_1x_2 \dots x_k1$, $x_i \in \{0, 2\}$ for $1 \leqslant i \leqslant k$. Suppose $z \in I$ and $z<x$. Then there exists $v \in I$, $z \leqslant v < x$ and where $v=0 \cdot x_1x_2 \dots x_k0y_1y_2 \dots y_p$, $y_i=2$ for $1 \leqslant i \leqslant p$ and for some $p \geqslant 1$. Let $u=0 \cdot x_1x_2 \dots x_k0y_1y_2 \dots y_{p-1}1$ then $[u, v] \in \mathscr{I}$ and it follows from the dual of (2) that $\underset{\sim}{c}=x=c$. If $c=y$ for some $[x, y] \in \mathscr{I}$, then $\underset{\sim}{c}=x<y=c$, so by the dual of (1), $\underset{\sim}{c}=c$. In order to prove in this case that $\tilde{c}=c$, note that $\tilde{c}=y=c$ so (2) must be used. Assume that $y=0 \cdot x_1x_2 \dots x_k2$, $x_i \in \{0, 2\}$ for $1 \leqslant i \leqslant k$. Suppose $z \in I$ and $y<z$. Then there exists $u \in I$, $y<u \leqslant z$ and where $u=0 \cdot x_1x_2 \dots x_k2y_1y_2 \dots \dots y_p1$, $y_i=0$ for $1 \leqslant i \leqslant p$ and for some p. Let $v=0 \cdot x_1x_2 \dots x_k2y_1y_2 \dots y_p2$ then $[u, v] \in \mathscr{I}$ and it follows that $\tilde{c}=c$. Finally, suppose $c \in I \sim \cup \mathscr{I}$ In this case $c=\underset{\sim}{c}=\tilde{c}$ so (2) and its dual apply. Since $c \in I \sim \cup \mathscr{I}$, c has an infinite expansion (i.e. it does not end in all 0's or in all 2's) and this expansion has only 0's and 2's as digits. Now suppose $z>c$. We must show that there exists an interval $[x, y] \in \mathscr{I}$ such that $c<x \leqslant z$. We may assume that $z \in I \sim \cup \mathscr{I}$, hence both c and z have an infinite expansion with only 0's and 2's as digits. Let $c=0 \cdot w_1w_2 \dots w_{k-1}w_k \dots$ and $z=0 \cdot w_1w_2 \dots \dots w_{k-1}z_k \dots$ where $w_k \neq z_k$. But $c<z$ so $w_k=0$ and $z_k=2$. Let $x=0 \cdot w_1w_2 \dots \dots w_{k-1}1$ and $y=0 \cdot w_1w_2 \dots w_{k-1}2$ then $c<x \leqslant z$ and $[x, y] \in \mathscr{I}$. It follows

that $c = \underset{\sim}{c}$. A similar argument shows that $c = \tilde{c}$. We conclude that $\theta^* = 0$ and therefore θ is not complemented.

4. COMPLETELY DISTRIBUTIVE COMPLETE LATTICES

We start this section with recalling some more definitions and results. If L is a lattice and $a, b \in L$, we write $a \prec b$ (or b *covers* a) if $a \leqslant b$ and $(a, b) = \emptyset$. L has the *jump property* if for all $a, b \in L$, $a < b$, there exist $c, d \in L$ such that $a \leqslant c \prec d \leqslant b$. L is *dense in itself* if for all $a, b \in L$ $a < b$, $(a, b) \neq \emptyset$. If $a, b \in L$ then b is an *immediate predecessor* of a if $b < a$ and $(a] = (b] \cup \{a\}$. An element a of a complete lattice L is *completely join irreducible* if $a \leqslant \sum S$, for $\emptyset \neq S \subseteqq L$ implies $a \leqslant s$ for some $s \in S$. Note that if L satisfies the infinite distributive law stated after Corollary 3.7 (and thus in particular, if L is completely distributive), then the condition of complete join irreducibility is equivalent to the following definition: $a \in L$ is completely join irreducible if $a = \sum S$, $\emptyset \neq S \subseteqq L$ implies $a = s$ for some $s \in S$. A *complete ring of sets* is a complete lattice whose elements are subsets of some set with set-theoretic operations as lattice operations (finite and infinite).

THEOREM 4.1. (Balachandran [1], Bruns [3], Raney [5]). Let L be a complete lattice. The following are equivalent: (i) L is a complete ring of sets; (ii) every element of L is the sum of completely join irreducible elements. If in addition, L is completely distributive, then (i) and (ii) are equivalent to: (iii) L has the jump property.

Before stating the next theorem, recall that a complete chain is completely distributive (cf. [2]) and also note that a non zero element of a complete chain is completely join irreducible if and only if it has an immediate predecessor.

THEOREM 4.2. (Raney [6], [7]). Let L be a complete lattice. The following are equivalent: (i) L is completely distributive; (ii) for $a, b \in L$, $a \nleqslant b$, there exist $p, q \in L$ such that $a \nleqslant p$, $b \ngeqslant q$ and $L = (p] \cup [q)$; (iii) L is a subdirect product of complete chains.

LEMMA 4.3. Let L be a complete lattice and let $a \in L$, $a \neq 0$. The following are equivalent: (i) a is completely join irreducible; (ii) there exists $a' \in L$, $a \nleqslant a'$ such that $L = (a'] \cup [a)$ (and moreover $(a'] \cap [a) = \emptyset$). If in addition, L is completely distributive, then (i) and (ii) are equivalent to: (iii) a has an immediate predecessor a_0.

PROOF. If a is completely join irreducible and $a' = \sum_{s \ngeqslant a} s$, then (ii) is immediate. That (ii) implies (i) is also immediate. If L is completely distributive and a is completely join irreducible, let $a_0 = \sum_{s < a} s$, then a_0 is an immediate predecessor of a.

(iii) $\Rightarrow$ (i) is trivial.

LEMMA 4.4. Let L, L' be complete lattices and let $h: L \to L'$ be a complete homomorphism which is onto. Suppose $b \in L'$, b completely join irreducible. Let $a = \prod \{x \in L; h(x) = b\}$. Then a is completely join irreducible and if $b \neq 0$, then $a \neq 0$.

PROOF. It is easy to see that $a = \prod \{x \in L: h(x) \geqslant b\}$ and that $h(a) = b$. If $b = 0$, then obviously $a = 0$. Thus assume $b \neq 0$. By Lemma 4.3, there exists $b' \in L'$ such that $b \nleqslant b'$ and $L' = (b'] \cup [b)$. Let $a' = \sum \{x \in L: h(x) \leqslant b'\}$. For $x \in L$, $h(x) \leqslant b'$ or $h(x) \geqslant b$, so $x \leqslant a'$ or $x \geqslant a$. Thus $L = (a'] \cup \cup [a)$. Also $a \nleqslant a'$ since $a \leqslant a'$ implies $h(a) \leqslant h(a')$ and thus $b \leqslant b'$. Finally, since $a \nleqslant a'$, $a \neq 0$. It follows from Lemma 4.3 that a is completely join irreducible.

LEMMA 4.5. Suppose L is a complete lattice which is a subdirect product of complete chains. Then L is dense in itself if and only if each chain is dense in itself.

PROOF. Suppose L is a subdirect product of complete chains $(C_i)_{i \in I}$. If for $i_0 \in I$, C_{i_0} is not dense in itself, then C_{i_0} has a non zero completely join irreducible element, but then by Lemma 4.4 L also has such an element. But since each C_i is completely distributive, L is completely distributive and it follows that L has an element which has an immediate predecessor and therefore, L is not dense in itself. Next, suppose that each C_i is dense in itself. Let for $x \in L$, x_i denote the projection of x on C_i. Suppose $a, b \in L$, $a < b$. There exists $i_0 \in I$ such that $a_{i_0} < b_{i_0}$. By hypothesis, there is an element $u \in C_{i_0}$ such that $a_{i_0} < u < b_{i_0}$ and an element $c \in L$ such that $c_{i_0} = u$. Since $(a + cb)_{i_0} = u$, we have $a < a + cb < b$ and thus L is dense in itself.

THEOREM 4.6. Let L be a completely distributive complete lattice. The following are equivalent: (i) L has no completely join irreducible elements except 0; (ii) L is dense in itself. Moreover if L is dense in itself then every complete homomorphic image of L is dense in itself.

PROOF. (i) $\Rightarrow$ (ii). By Theorem 4.2, L is a subdirect product of complete chains. By Lemma 4.4, each of these chains have no completely join irreducible elements except 0. Therefore each chain is dense in itself and thus by Lemma 4.5, L is dense in itself. (ii) $\Rightarrow$ (i). Immediate from Lemma 4.3. Finally, suppose L is dense in itself and L' is a complete homomorphic image of L which is not dense in itself. Then by (i) $\Rightarrow$ (ii) of this theorem, L' has a completely join irreducible element $\neq 0$ and it follows from Lemma 4.4, that L has such an element. But then again by (ii) $\Rightarrow$ (i) of this theorem, L is not dense in itself. Contradiction. This completes the proof of the theorem.

THEOREM 4.7. Let L be a completely distributive complete lattice which is not a complete ring of sets. Then L has a complete homomorphic image which is not $\mathbf{1}$ and which is dense in itself (and which is therefore infinite).

PROOF. By Theorem 4.1, there exists an element $a \in L$, $a \neq 0$ which is not the sum of completely join irreducible elements. Let $L_1 = (a]$. It is obvious that an element of L_1 is completely join irreducible in L_1 if and only if it is completely join irreducible in L. Also note that $L_1 \neq \mathbf{1}$. Let $a_0 = \sum \{x \in L_1 : x \text{ completely join irreducible in } L_1\}$. Then $a_0 < 1_{L_1}(=a)$. If $h: L \to L_1$ is defined by $h(x) = ax$ for $x \in L$ then obviously, h is a complete homomorphism which is onto. Let θ be the congruence relation on L_1 defined by $x \equiv y(\theta) \Leftrightarrow x + a_0 = y + a_0$, for $x, y \in L_1$. Then θ is complete and therefore the homomorphism $h_1: L_1 \to L_{1/\theta}$ is complete (and onto). Since obviously $a \not\equiv a_0(\theta)$, we have that $L_{1/\theta} \neq \mathbf{1}$. We show that 0 is the only completely join irreducible element of $L_{1/\theta}$. Indeed, suppose $b \in L_{1/\theta}$, $b \neq 0$ and b completely join irreducible. Then by Lemma 4.4, there exists an element $a_1 \in L_1$, $a_1 \neq 0$, $h_1(a) = b$ and a_1 completely join irreducible. But then $a_1 \leqslant a_0$ and therefore $h_1(a_1) = b \leqslant h(a_0) = 0$. Contradiction. By theorem 4.6, $L_{1/\theta}$ is therefore dense in itself. Since $L_{1/\theta}$ is a complete homomorphic image of L and $L_{1/\theta} \neq \mathbf{1}$, the proof is complete.

COROLLARY 4.8. Let L be a completely distributive complete lattice which is not a complete ring of sets. Then L has a complete homomorphic image which is not $\mathbf{1}$ and which is a complete dense in itself chain.

PROOF. By Theorem 4.7, L has a complete homomorphic image $L_1 \neq \mathbf{1}$ which is dense in itself. But by Theorem 4.2, L_1 is a subdirect product of complete chains. Since $L_1 \neq \mathbf{1}$, at least one of those chains is not $\mathbf{1}$ and this chain is by Lemma 4.5, (or by Theorem 4.6) dense in itself.

5. CLASSES OF COMPLETELY DISTRIBUTIVE COMPLETE LATTICES

In this section we will investigate two special classes of completely distributive complete lattices. The first of these is the class $\mathscr{R}$ consisting of all completely distributive complete lattices all of whose complete homomorphic images are complete rings of sets. We have already observed in section 1, that complete atomic Boolean algebras and complete (dual) ordinals belong to $\mathscr{R}$. The first of these facts is of course known and the second follows immediately from Theorem 4.1. We will now show first that every countable, completely distributive complete lattice also belongs to $\mathscr{R}$.

THEOREM 5.1. Let L be a completely distributive complete lattice which is countable. Then L belongs to $\mathscr{R}$.

PROOF. It suffices of course to show that every completely distributive complete lattices which is countable is a complete ring of sets. We first prove this for complete chains. Thus, suppose that C is a countable complete chain and assume that C is not a complete ring of sets. Then by Theorem 4.1, C has not the jump property. Therefore C has an interval $[a, b]$, $a<b$, which is dense in itself. But $[a, b]$ is countable and therefore $[a, b]$ is isomorphic to the unit interval of the rationals which however, is not complete. It follows that C is a complete ring of sets. The general case follows now immediately from Theorem 4.2.

Our main result concerning the characterization of $\mathscr{R}$ is contained in the following theorem.

THEOREM 5.2. Let L be a completely distributive complete lattice. Then $L \in \mathscr{R}$ if and only if $\mathrm{Con}(L)$ is completely distributive. Moreover, if $L \in \mathscr{R}$ then $\mathrm{Con}(L)$ is isomorphic to 2^X for some set X.

We will prove a sequence of lemmas which will lead to the proof of the theorem.

LEMMA 5.3. Let C be a complete chain which is dense in itself and $C \neq 1$. Suppose $\theta \in \mathrm{Con}(C)$. Then we have: (i) if $\theta \neq 0$, there exists $\theta_1 \in \mathrm{Con}(C)$ such that $0<\theta_1<\theta$; (ii) if $\theta \neq 1$, then there exists $\theta_1 \in \mathrm{Con}(C)$ such that $\theta<\theta_1<1$; if $\theta \neq 0, 1$, then there exists $\theta_1 \in \mathrm{Con}(C)$ such that $\theta_1 \not\leq \theta$ and $\theta_1 \not\geq \theta$.

PROOF (i). By hypothesis, θ has a congruence class $[a, b]$ such that $a<b$. By density of C, there exists an element $a' \in C$, such that $a<a'<b$. Let $\theta_1 \in \mathrm{Con}(C)$ be defined by $[a, a']$ and the intervals $[x, x]$, $x \in C \sim [a, a']$. Obviously, $0<\theta_1<\theta$. (ii) θ has a congruence class $[a, b] \neq C$, thus either $0<a$ or $b<1$, say, $0<a$. If $[0]_\theta = [0, c]$ then $c<a$. By density of C, there exists $d \in C$ such that $c<d<a$. If $[d]_\theta = [d_1, d_2]$, then $c<d_1<d<d_2<a$. Let $\theta_1 \in \mathrm{Con}(C)$ be defined by $[d_1, b]$ and the intervals $[x]_\theta$ for $x \in C \sim [d_1, b]$. Then $\theta<\theta_1<1$. (iii) Since $0<\theta<1$, θ has a congruence class $[a, b]$ such that $a<b$ and $[a, b] \neq C$, say $a \neq 0$. By density of C there exist $c \in C$ such that $0<c<a$ and $d \in C$ such that $a<d<b$. Let $\theta_1 \in \mathrm{Con}(C)$ be defined by $[c, d]$ and the intervals $[x, x]$, $x \in C \sim [c, d]$. Obviously $\theta_1 \not\leq \theta$ and $\theta_1 \not\geq \theta$.

LEMMA 5.4. Let C be a complete chain which is dense in itself, $C \neq 1$. Then $\mathrm{Con}(C)$ is not completely distributive.

PROOF. In this proof we will without danger of confusion, denote both the zero (one) element of C and of $\mathrm{Con}(C)$ by $0(1)$. Also note, that since $C \neq 1$, we have that $\mathrm{Con}(C) \neq 1$. We will show that $\mathrm{Con}(C)$ does not satisfy condition (ii) of Theorem 4.2. Suppose there exist $\theta_1, \theta_2 \in \mathrm{Con}(C)$, such that $1 \not\leq \theta_1$, $0 \not\geq \theta_2$ and such $\mathrm{Con}(C) = (\theta_1] \cup [\theta_2)$. Now $0 \neq \theta_1$, since

$0=\theta_1$ implies Con $(C)=(0]\cup[\theta_2)$. But $\theta_2\neq 0$ so by Lemma 5.3, there exists $\theta'\in$ Con (C) such that $0<\theta'<\theta_2$ and then $\theta'\notin(0]\cup[\theta_2)$, Contradiction. Hence $0<\theta_1<1$. Similarly, again applying Lemma 5.3, it follows that $\theta_2\neq 1$ and thus $0<\theta_2<1$. We will now show that there exists $\theta\in$Con (C) such that $\theta\notin(\theta_1]\cup[\theta_2)$. Since $0<\theta_1<1$ and $0<\theta_2<1$, θ_1 has a congruence class $[a_1, b_1]$ and θ_2 has a congruence relation $[a_2, b_2]$, such that $a_1<b_1$, $[a_1, b_1]\neq C$, $a_2<b_2$ and $[a_2, b_2]\neq C$. There are two cases:

(i) $[a_2, b_2]\subseteq[a_1, b_1]$. Since $[a_1, b_1]\neq C$ either $0<a_1$ or $b_1<1$, say $b_1<1$. Let $\theta\in$ Con (C) be defined by the closed intervals $[b_1, 1]$ and $[x, x]$, $x\in C\sim[b_1, 1]$. Now $b_1\equiv 1(\theta)$ but $b_1\not\equiv 1(\theta_1)$ so $\theta\nleqslant\theta_1$. Again, $a_2\equiv b_2(\theta_2)$ but $a_2\neq b_2\leqslant b_1$ so $a_2\not\equiv b_2(\theta)$ and thus $\theta_2\nleqslant\theta$. Hence $\theta\notin(\theta_1]\cup[\theta_2)$.

(ii) $[a_2, b_2]\nsubseteq[a_1, b_1]$. Then either $b_2>b_1$ or $a_1<a_2$, say $b_2>b_1$. Let $a=a_2+b_1$. But $a_2<b_2$ and $b_1<b_2$, thus $a<b_2$. By density of C, there exists $c\in C$, such that $a<c<b_2$. Thus $b_1\leqslant a<c<b_2$. So $b_1\not\equiv c(\theta_1)$ since $b_1\equiv c(\theta_1)\Rightarrow c\in[a_1, b_1]\Rightarrow c\leqslant b_1$, but $c>b_1$. Let $\theta\in$ Con (C) be defined by the closed intervals $[b_1, c]$ and $[x, x]$, $x\in C\sim[b_1, c]$. So $b_1\equiv c(\theta)$ and thus $\theta\nleqslant\theta_1$. Again, we have $a_2\leqslant a<c<b_2$. Therefore $c\equiv b_2(\theta_2)$. But also $b_1<c<b_2$ so $b_2\notin[b_1, c]$ and thus $b_2\not\equiv c(\theta)$. Therefore $\theta_2\nleqslant\theta$. It follows that $\theta\notin(\theta_1]\cup[\theta_2)$, completing the proof of the lemma.

LEMMA 5.5. *Let L be a completely distributive complete lattice such that* Con (C) *is completely distributive. Then L is a complete ring of sets.*

PROOF. Suppose L is not a complete ring of sets. By Corollary 4.8, there exists a complete onto homomorphism $h\colon L\to C$, where C is a complete chain which is dense in itself and $C\neq 1$. By Lemma 5.4, Con (C) is not completely distributive. Let θ be the kernel of h. Then Con $(C)\cong[\theta)_{\mathrm{Con}\,(L)}$. Thus $[\theta)_{\mathrm{Con}\,(L)}$ and therefore Con (L), is not completely distributive.

Before we will state the next lemma, we will introduce the following useful notation. Let L be a complete lattice. Then $\mathrm{Con}_0\,(L)=\{\theta\in\mathrm{Con}\,(L)$: $L/_\theta$ is a complete ring of sets$\}$. $\mathrm{Con}_0\,(L)$ is considered as a partially ordered set under the same partial ordering as that of Con (L).

LEMMA 5.6. *Let L be a complete ring of sets and let X be the set of completely non zero join irreducible elements of L. Then* $\mathrm{Con}_0\,(L)\cong 2^X$.

PROOF. By Theorem 4.1, if $a\in L$, then $a=\sum\{x\colon x\in X, x\leqslant a\}$. Let $\hat{a}=\{x\colon x\in X, x\leqslant a\}$ then the map $a\mapsto\hat{a}$ is a regular embedding (i.e. a one – one complete homomorphism) of L into 2^X (cf. [2]). We now define for every $X_1\subseteq X$ a map $h_{X_1}\colon L\to 2^X$, by $h_{X_1}(a)=\hat{a}\cap X_1$. Obviously, h_{X_1} is a complete homomorphism and $h_{X_1}[L]$ is a complete ring of sets. Therefore $\theta_{X_1}=$kernel $h_{X_1}\in\mathrm{Con}_0\,(L)$. The map $X_1\mapsto\theta_{X_1}$ for $X_1\subseteq X$ establishes a map $2^X\to\mathrm{Con}_0\,(L)$. We now proceed in steps and show:

(i) the map is onto. Let $\theta_0\in\mathrm{Con}_0\,(L)$ and let $h\colon L\to L/_{\theta_0}$ be the canonical homomorphism. Let Y_0 be the set of completely non zero join irreducible elements of $L/_{\theta_0}$ and let

$$X_0 = \{ \prod_{h(x)=y} z : y \in Y_0\} = \{ \prod_{h(z)\geqslant y} z : y \in Y_0\}.$$

Then by Lemma 4.4, $X_0 \subseteq X$. We claim that $\theta_{X_0} = \theta_0$. It suffices to show that $h(a)=h(b) \Leftrightarrow \hat{a} \cap X_0 = \hat{b} \cap X_0$. Suppose first, $h(a)=h(b)$ and let $x_0 \in \hat{a} \cap X_0$. Thus

$$x_0 = \prod_{h(z)=y_0} z = \prod_{a(z)\geqslant y_0} z \text{ for some } y_0 \in Y.$$

We have $h(x_0)=y_0$. But also, $x_0 \leqslant a \Rightarrow y_0 = h(x_0) \leqslant h(a) = h(b) \Rightarrow y_0 \leqslant h(b) \Rightarrow$ $\Rightarrow x_0 \leqslant b \Rightarrow x \in \hat{b} \Rightarrow x_0 \in \hat{b} \cap X_0$. Next, suppose $\hat{a} \cap X_0 = \hat{b} \cap X_0$. We must show that $h(a)=h(b)$ that is, we must show that $Y_0 \cap (h(a)] = Y_0 \cap (h(b)]$. Suppose $y_0 \in Y_0 \cap (h(a)]$. Let

$$x_0 = \prod_{h(z) > y_0} z, \text{ then } x_0 \in X \text{ and } h(x_0) = y_0.$$

Now

$$y_0 \leqslant h(a) \Rightarrow x_0 \leqslant a \Rightarrow x_0 \in \hat{a} \cap X_0 \Rightarrow x_0 \in \hat{b} \cap X_0 \Rightarrow x_0 \leqslant b \Rightarrow$$
$$\Rightarrow h(x_0) = y_0 \leqslant h(b) \Rightarrow y_0 \in Y_0 \cap (h(b)].$$

Hence $Y_0 \cap (h(a)] \subseteq Y_0 \cap (h(b)]$. The reverse inequality follows similarly.

(ii) $X_1 \subseteq X_2 \Rightarrow \theta_{X_1} \geqslant \theta_{X_2}$ for $X_1, X_2 \subseteq X$. This is obvious.

(iii) $\theta_{X_1} \geqslant \theta_{X_2} \Rightarrow X_1 \subseteqq X_2$ for $X_1, X_2 \subseteq X$. Suppose $X_1 \nsubseteq X_2$. Then there exists $x \in X_1$, $x \notin X_2$. Since $x \neq 0$, x has by Lemma 4.3 an immediate predecessor x_0. We first show that $x_0 \equiv x(\theta_{X_2})$. Thus, to show that $\hat{x}_0 \cap X_2 = \hat{x} \cap X_2$. Now

$$y \in \hat{x}_0 \cap X_2 \Rightarrow y \leqslant x_0, y \in X_2 \Rightarrow y \leqslant x, y \in X_2 \Rightarrow y \in \hat{x} \cap X_2.$$

Again, $y \in \hat{x} \cap X_2 \Rightarrow y \leqslant x, y \in X_2$. But $y \neq x$ since $y = x$ would imply $x \in X_2$. Hence $y < x$ and therefore $y \leqslant x_0$ and it follows that $y \in \hat{x}_0 \cap X_2$. Next we show that $x_0 \not\equiv x(\theta_{X_1})$. Thus we must show that $\hat{x}_0 \cap X_1 \neq \hat{x} \cap X_1$. Now $x_0 < x \Rightarrow x \notin \hat{x}_0 \cap X_1$. Again, $x \leqslant x, x \in X_1 \Rightarrow x \in \hat{x} \cap X_1$. Hence $\hat{x} \cap X_1 \nsubseteq \hat{x}_0 \cap X_1$. It follows that $\theta_{X_1} \ngeqslant \theta_{X_2}$ completing the proof of (iii).

It now follows that the map $X_1 \mapsto \theta_{X_1}$ for $X_1 \subseteqq X$, establishes an isomorphism between the dual algebra of 2^X and $\mathrm{Con}_0(L)$. But a Boolean algebra is isomorphic to its dual and therefore $\mathrm{Con}_0(L)$ and 2^X are isomorphic.

We are now ready to prove Theorem 5.2.

PROOF OF THEOREM 5.2. First, suppose $L \notin \mathscr{R}$. Then there exists $\theta \in \mathrm{Con}(L)$ such that $L/_\theta$ is not a complete ring of sets. By Lemma 5.5, $\mathrm{Con}(L/_\theta)$ is not completely distributive. But $\mathrm{Con}(L/_\theta) = [\theta)_{\mathrm{Con}(L)}$ and thus $\mathrm{Con}(L)$ is not completely distributive. Next, assume that $L \in \mathscr{R}$. But then by Lemma 5.6, Con $(L)=\mathbf{2}^X$ where X is the set of completely non zero join irreducible elements of L.

We finally consider the class $\mathscr{D}$ of all completely distributive complete lattices none of whose complete homomorphic images (except $\mathbf{1}$) is a complete ring of sets. The following theorem characterizes the class $\mathscr{D}$.

THEOREM 5.7. Suppose L is a completely distributive complete lattice. Then $L \in \mathscr{D}$ if and only if L is dense in itself.

PROOF. Suppose L is dense in itself. Then by Theorem 4.6, L has no completely join irreducible elements except 0. If L' is a complete homomorphic image of L, then it follows from Lemma 4.4 that L' has no completely join irreducible elements except 0. If in addition, $L' \neq \mathbf{1}$, then we infer from Theorem 4.1 that L' is not a complete ring of sets. Next, suppose that L is not dense in itself. Then by Theorem 4.6, L has an element a, $a \neq 0$ such that a is completely join irreducible. But then the map $h: L \rightarrow \mathbf{2}$, defined by $h(x)=0 \Leftrightarrow x \ngeq a$ is a complete onto homomorphism. Hence L has a complete homomorphic image which is a complete ring of sets and which is not $\mathbf{1}$. This completes the proof of the theorem.

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